



A regularity criterion for the three-dimensional micropolar fluid system in homogeneous Besov spaces

Zujin Zhang 

School of Mathematics and Computer Sciences, Gannan Normal University
Ganzhou 341000, Jiangxi, P.R. China

Received 7 September 2016, appeared 21 November 2016

Communicated by Maria Alessandra Ragusa

Abstract. By establishing a new trilinear estimate, we show a regularity criterion for the three dimensional micropolar fluid system via the velocity in homogeneous Besov spaces. This improves [B. Q. Dong, Z. L. Zhang, On the regularity criterion for three-dimensional micropolar fluid flows in Besov spaces, *Nonlinear Anal.* 73(2010), 2334–2341] in some sense.

Keywords: micropolar fluid flows, regularity criteria, Besov spaces.

2010 Mathematics Subject Classification: 35Q35, 35B44.


1 Introduction

In this paper, we study the following three-dimensional micropolar fluid system with unit viscosities:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} - \Delta \mathbf{u} + \nabla P - \nabla \times \mathbf{w} = \mathbf{0}, \\ \partial_t \mathbf{w} + (\mathbf{u} \cdot \nabla) \mathbf{w} - \Delta \mathbf{w} + 2\mathbf{w} - \nabla \operatorname{div} \mathbf{w} - \nabla \times \mathbf{u} = \mathbf{0}, \\ \nabla \cdot \mathbf{u} = 0, \\ (\mathbf{u}, \mathbf{w})|_{t=0} = (\mathbf{u}_0, \mathbf{w}_0), \end{cases} \quad (1.1)$$

where $\mathbf{u} = (u_1, u_2, u_3)$ is the fluid velocity field, P is the pressure arising from the usual assumption of incompressibility $\operatorname{div} \mathbf{u} = 0$, $\mathbf{w} = (w_1, w_2, w_3)$ is the micro-rotation vector field, \mathbf{u}_0 and \mathbf{w}_0 are the prescribed initial data satisfying $\nabla \cdot \mathbf{u}_0 = 0$.

The modern theory of micropolar fluid system was initialled by Eringen [9], which models some physical phenomena that cannot be treated by the classical Navier–Stokes equations, for example, the motion of animal blood, liquid crystals and dilute aqueous polymer solutions for example. Mathematically, many authors [3, 4, 7, 10, 20, 22, 24] treated the well-posedness and large-time behaviour of solutions to system (1.1). However, the issue of global regularity of weak solutions to (1.1) remains an open problem. Therefore it is important to study the

 Email: zhangzujin361@163.com

regularity criterion on the some physical quantities, such as velocity, vorticity and pressure. Dong and Zhang [6] showed the following regularity condition

$$\mathbf{u} \in L^{\frac{2}{1+r}}(0, T; B_{\infty, \infty}^r(\mathbb{R}^3)), \quad -1 < r < 1. \quad (1.2)$$

Here and in what follows, $\dot{B}_{p, q}^s(\mathbb{R}^3)$ (resp. $B_{p, q}^s(\mathbb{R}^3)$) with $s \in \mathbb{R}$, $p, q \in [1, \infty]$ is the inhomogeneous (homogeneous) Besov space, whose definition, fine properties and its utilization in fluid dynamical systems can be found in [1]. Later, Wang and Yuan [23] and He and Wang [17] (with [17, Equation 1.10] replaced by [26]) then considered the following regularity criterion in a logarithmically improved version

$$\int_0^T \frac{\|P\|_{\dot{B}_{\infty, \infty}^r}^2}{1 + \ln(e + \|P\|_{\dot{B}_{\infty, \infty}^r})} dt < \infty, \quad -1 \leq r < 1. \quad (1.3)$$

For interested readers, please refer to [5, 8, 11–16, 18, 25, 29].

The purpose of this paper is to improve (1.2) from inhomogeneous Besov spaces to homogeneous ones. Before stating the precise result, let us recall the weak formulation of (1.1).

Definition 1.1. Let $\mathbf{u}_0 \in L^2(\mathbb{R}^3)$ with $\operatorname{div} \mathbf{u}_0 = 0$, $\mathbf{w}_0 \in L^2(\mathbb{R}^3)$. A measurable \mathbb{R}^3 -valued pair (\mathbf{u}, \mathbf{w}) is called a weak solution to system (1.1) on $(0, T)$, provided the following two conditions hold,

- (1) $(\mathbf{u}, \mathbf{w}) \in L^\infty(0, T; L^2(\mathbb{R}^3)) \cap L^2(0, T; H^1(\mathbb{R}^3));$
- (2) (\mathbf{u}, \mathbf{w}) verifies (1.1) in the distributional sense.

Now, our result reads as follows.

Theorem 1.2. Let $T > 0$, $0 < r < 1$ and (\mathbf{u}, \mathbf{w}) be a weak solution pair of system (1.1) with initial data $(\mathbf{u}_0, \mathbf{w}_0) \in H^1(\mathbb{R}^3)$. If

$$\mathbf{u} \in L^{\frac{2}{1+r}}(0, T; \dot{B}_{\infty, \infty}^r(\mathbb{R}^3)), \quad (1.4)$$

then the solution (\mathbf{u}, \mathbf{w}) is regular on $(0, T)$.

Remark 1.3. Due to the fact that $B_{p, q}^s(\mathbb{R}^3) = L^p(\mathbb{R}^3) \cap \dot{B}_{p, q}^s(\mathbb{R}^3)$ for any $s > 0$ and $1 \leq p, q \leq \infty$ (see [2, Theorem 6.3.2]), we point that, in the case $0 < r < 1$, (1.4) is an improvement of (1.2).

Remark 1.4. The refinement $\mathbf{u} \in L^1(0, T; \dot{B}_{\infty, \infty}^1(\mathbb{R}^3))$ of (1.2) in case $r = 1$ was already established in [5, Theorem 1.3]; while the gain

$$\mathbf{u} \in L^2(0, T; \dot{B}_{\infty, \infty}^0(\mathbb{R}^3)) \quad (1.5)$$

of (1.2) in case $r = 0$ can be similarly verified just as [27, Theorem 1.1].

Remark 1.5. By observing the following fact from [21, Section 1.3]:

$$s < 0, p, q \geq 1 \Rightarrow \dot{B}_{p, q}^s(\mathbb{R}^3) \subset B_{p, q}^s(\mathbb{R}^3),$$

we see that the condition $\mathbf{u} \in L^{\frac{2}{1+r}}(0, T; \dot{B}_{\infty, \infty}^r(\mathbb{R}^3))$ with $-1 < r < 0$ could also ensure the smoothness of the solution. One is referred to [19, 28] for similar results of the Navier–Stokes equations.

Before showing Theorem 1.2 in Section 2, let us end this introduction by proving the following trilinear estimate, which could have its own interest.

Lemma 1.6. *For $f \in \dot{B}_{\infty,\infty}^r(\mathbb{R}^3)$, $g, h \in H^1(\mathbb{R}^3)$ and any $\varepsilon > 0$, $0 < r < 1$, $k \in \{1, 2, 3\}$, we have*

$$\int_{\mathbb{R}^3} \partial_k f \cdot gh \, dx \leq C \|f\|_{\dot{B}_{\infty,\infty}^r}^{\frac{2}{1+r}} \|(g, h)\|_{L^2}^2 + \varepsilon \|\nabla(g, h)\|_{L^2}^2. \quad (1.6)$$

Proof.

$$\begin{aligned} \int_{\mathbb{R}^3} \partial_k f \cdot gh \, dx &= - \int_{\mathbb{R}^3} f \cdot \partial_k(gh) \, dx = - \int_{\mathbb{R}^3} \Lambda^r f \cdot \Lambda^{-r} \partial_k(gh) \, dx \quad (\Lambda = (-\Delta)^{\frac{1}{2}}) \\ &\leq C \|\Lambda^r f\|_{\dot{B}_{\infty,\infty}^0} \|\Lambda^{-r} \partial_k(gh)\|_{\dot{B}_{1,1}^0} \quad (\text{by [1, Proposition 2.29]}) \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r} \|gh\|_{\dot{B}_{1,1}^{1-r}} \quad (\text{by [1, Lemma 2.1]}) \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r} \left(\|g\|_{L^2} \|h\|_{\dot{B}_{2,1}^{1-r}} + \|g\|_{\dot{B}_{2,1}^{1-r}} \|h\|_{L^2} \right) \\ &\quad (\text{by analogues of [1, Corollary 2.54]}) \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r} \left(\|g\|_{L^2} \|h\|_{\dot{B}_{2,\infty}^0}^r \|h\|_{\dot{B}_{2,\infty}^{1-r}}^{1-r} + \|g\|_{\dot{B}_{2,\infty}^0}^r \|g\|_{\dot{B}_{2,\infty}^{1-r}}^{1-r} \|h\|_{L^2} \right) \\ &\quad (\text{by [1, Proposition 2.22]}) \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r} \left(\|g\|_{L^2} \|h\|_{L^2}^r \|\nabla h\|_{L^2}^{1-r} + \|g\|_{L^2}^r \|\nabla g\|_{L^2}^{1-r} \|h\|_{L^2} \right) \\ &\quad (\text{by [1, Proposition 2.39]}) \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r} \|(g, h)\|_{L^2}^{1+r} \|\nabla(g, h)\|_{L^2}^{1-r} \\ &\leq C \|f\|_{\dot{B}_{\infty,\infty}^r}^{\frac{2}{1+r}} \|(g, h)\|_{L^2}^2 + \varepsilon \|\nabla(g, h)\|_{L^2}^2. \end{aligned}$$

□

2 Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2. Taking the inner product of (1.1)₁ with $-\Delta \mathbf{u}$ in $L^2(\mathbb{R}^3)$, we obtain

$$\frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{u}\|_{L^2}^2 + \|\Delta \mathbf{u}\|_{L^2}^2 = \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \Delta \mathbf{u} \, dx - \int_{\mathbb{R}^3} (\nabla \times \mathbf{w}) \cdot \Delta \mathbf{u} \, dx. \quad (2.1)$$

On the other hand, testing (1.1)₂ by $-\Delta \mathbf{w}$, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla \mathbf{w}\|_{L^2}^2 + \|\Delta \mathbf{w}\|_{L^2}^2 + 2 \|\nabla \mathbf{w}\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{w}\|_{L^2}^2 \\ = \int_{\mathbb{R}^3} [(\mathbf{u} \cdot \nabla) \mathbf{w}] \cdot \Delta \mathbf{w} \, dx - \int_{\mathbb{R}^3} (\nabla \times \mathbf{u}) \cdot \Delta \mathbf{w} \, dx. \end{aligned} \quad (2.2)$$

Plugging (2.1) and (2.2) together, integrating by parts then yields

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|\nabla(\mathbf{u}, \mathbf{w})\|_{L^2}^2 + \|\Delta(\mathbf{u}, \mathbf{w})\|_{L^2}^2 + 2 \|\nabla \mathbf{w}\|_{L^2}^2 + \|\nabla \operatorname{div} \mathbf{w}\|_{L^2}^2 \\ = \left[- \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i \mathbf{u} \cdot \nabla) \mathbf{u}] \cdot \partial_i \mathbf{u} \, dx - \sum_{i=1}^3 \int_{\mathbb{R}^3} [(\partial_i \mathbf{u} \cdot \nabla) \mathbf{w}] \cdot \partial_i \mathbf{w} \, dx \right] \\ - 2 \int_{\mathbb{R}^3} (\nabla \times \mathbf{w}) \cdot \Delta \mathbf{u} \, dx \\ \equiv I + J. \end{aligned} \quad (2.3)$$

By Lemma 1.6,

$$I \leq C \|u\|_{B_{\infty,\infty}^{\frac{2}{1+r}}}^{\frac{2}{1+r}} \|\nabla(u, w)\|_{L^2}^2 + \frac{1}{4} \|\Delta(u, w)\|_{L^2}^2. \quad (2.4)$$

Meanwhile,

$$J \leq 2 \|\nabla \times w\|_{L^2} \|\Delta u\|_{L^2} \leq 2 \|\nabla w\|_{L^2}^2 + \frac{1}{2} \|\Delta u\|_{L^2}^2. \quad (2.5)$$

Putting (2.4) and (2.5) into (2.3), we deduce

$$\frac{1}{2} \frac{d}{dt} \|\nabla(u, w)\|_{L^2}^2 + \frac{1}{4} \|\Delta(u, w)\|_{L^2}^2 + \|\nabla \operatorname{div} w\|_{L^2}^2 \leq C \|u\|_{B_{\infty,\infty}^{\frac{2}{1+r}}}^{\frac{2}{1+r}} \|\nabla(u, w)\|_{L^2}^2.$$

Applying Gronwall's inequality with the fact (1.4), we find

$$\|\nabla(u, w)(t)\|_{L^2}^2 \leq \|\nabla(u_0, w_0)\|_{L^2}^2 \cdot \exp \left[\int_0^t \|u(\tau)\|_{B_{\infty,\infty}^{\frac{2}{1+r}}}^{\frac{2}{1+r}} d\tau \right] < \infty, \quad \forall 0 \leq t < T.$$

By Sobolev's inequality,

$$u \in L^\infty(0, T; L^6(\mathbb{R}^3)) \subset L^4(0, T; L^6(\mathbb{R}^3)).$$

By [5, Theorem 1.1], we complete the proof of Theorem 1.2.

Acknowledgements

This work is supported by the Natural Science Foundation of Jiangxi (grant no. 20151BAB201010), the National Natural Science Foundation of China (grant nos. 11501125, 11361004) and the Supporting the Development for Local Colleges and Universities Foundation of China – Applied Mathematics Innovative Team Building.

References

- [1] H. BAHOURI, J. Y. CHEMIN, R. DANCHIN, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften, Vol. 343, Springer, Heidelberg, 2011. [MR2768550](#); [url](#)
- [2] J. BERGH, J. LÖFSTRÖM, *Interpolation spaces: an introduction*, Grundlehren der Mathematischen Wissenschaften, Vol. 223, Springer-Verlag, Berlin, New York, Heidelberg, 1976. [MR0482275](#); [url](#)
- [3] J. W. CHEN, Z.M. CHEN, B.Q. DONG, Uniform attractors of non-homogeneous micropolar fluid flows in non-smooth domains, *Nonlinearity* **20**(2007), 1619–1635. [MR2335076](#); [url](#)
- [4] B. Q. DONG, Z. M. CHEN, Asymptotic profiles of solutions to the 2D viscous incompressible micropolar fluid flows, *Discrete Contin. Dyn. Syst.* **23**(2009), 765–784. [MR2461826](#); [url](#)
- [5] B. Q. DONG, Z. M. CHEN, Regularity criteria of weak solutions to the three-dimensional micropolar flows, *J. Math. Phys.* **50**(2009), 103525, 13 pp. [MR2572698](#); [url](#)
- [6] B. Q. DONG, Z. L. ZHANG, On the regularity criterion for three-dimensional micropolar fluid flows in Besov spaces, *Nonlinear Anal.* **73**(2010), 2334–2341. [MR2674209](#); [url](#)

- [7] B. Q. DONG, Z. F. ZHANG, Global regularity of the 2D micropolar fluid flows with zero angular viscosity, *J. Differential Equations* **249**(2010), 200–213. [MR2644133](#); [url](#)
- [8] L. L. DU, D. Q. ZHOU, Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion, *SIAM J. Math. Anal.* **47**(2015), 1562–1589. [MR3338000](#); [url](#)
- [9] A. C. ERINGEN, Theory of micropolar fluids, *J. Math. Mech.* **16**(1966), 1–18. [MR0204005](#)
- [10] G. P. GALDI, S. RIONERO, A note on the existence and uniqueness of solutions of the micropolar fluid equations, *Internat. J. Engrg. Sci.* **15**(1977), 105–108. [MR0467030](#); [url](#)
- [11] S. GALA, On regularity criteria for the three-dimensional micropolar fluid equations in the critical Morrey–Campanato space, *Nonlinear Anal. Real World Appl.* **12**(2011), 2142–2150. [MR2801007](#); [url](#)
- [12] S. GALA, Regularity criteria for the 3D magneto-micropolar fluid equations in the Morrey–Campanato space, *Nonlinear Differ. Equ. Appl.* **17**(2010), 181–194. [MR2639150](#); [url](#)
- [13] S. GALA, M. A. RAGUSA, A logarithmic regularity criterion for the two-dimensional MHD equations, *J. Math. Anal. Appl.* **444**(2016), 1752–1758. [MR3535787](#); [url](#)
- [14] S. GALA, M. A. RAGUSA, A regularity criterion for 3D micropolar fluid flows in terms of one partial derivative of the velocity, *Ann. Polon. Math.* **116**(2016), 217–228. [MR3506781](#); [url](#)
- [15] S. GALA, M. A. RAGUSA, Note on the weak-strong uniqueness criterion for the β -QG in Morrey–Campanato space, *Appl. Math. Comput.* **293**(2017), 65–71. [MR3549653](#); [url](#)
- [16] C. C. GUO, Z. J. ZHANG, J. L. WANG, Regularity criteria for the 3D magneto-micropolar fluid equations in Besov spaces with negative indices, *Appl. Math. Comput.* **218**(2012), 10755–10758. [MR2927090](#); [url](#)
- [17] X. W. HE, S. GALA, Regularity criterion for weak solutions to the Navier–Stokes equations in terms of the pressure in the class $L^2(0, T; \dot{B}_{\infty, \infty}^{-1}(\mathbb{R}^3))$, *Nonlinear Anal. Real World Appl.* **12**(2011), 3602–3607. [MR2832995](#); [url](#)
- [18] Y. JIA, X. W. ZHANG, W. L. ZHANG, B. Q. DONG, Remarks on the regularity criteria of weak solutions to the three-dimensional micropolar fluid equations, *Acta Math. Appl. Sin. Engl. Ser.* **29**(2013), 869–880. [MR3133825](#); [url](#)
- [19] H. KOZONO, Y. SHIMADA, Bilinear estimates in homogeneous Triebel–Lizorkin spaces and the Navier–Stokes equations, *Math. Nachr.* **276**(2004), 63–74. [MR2100048](#); [url](#)
- [20] G. LUKASZEWICZ, *Micropolar fluids. Theory and applications*, Modeling and Simulation in Science, Engineering and Technology, Birkhäuser Boston, Inc., Boston, MA, 1999. [MR1711268](#); [url](#)
- [21] C. X. MIAO, J. H. WU, Z. F. ZHANG, *Littlewood–Paley theory and its application in hydrodynamic equations* (Chinese Edition), Science Press, 2012.
- [22] M. A. ROJAS-MEDAR, Magneto-micropolar fluid motion: existence and uniqueness of strong solution, *Math. Nachr.* **188**(1997), 301–319. [MR1484679](#); [url](#)

- [23] Y. Z. WANG, H. C. YUAN, A logarithmically improved blow-up criterion for smooth solutions to the 3D micropolar fluid equations, *Nonlinear Anal. Real World Appl.* **13**(2012), 1904–1912. [MR2891019](#); [url](#)
- [24] K. YAMAZAKI, Global regularity of the two-dimensional magneto-micropolar fluid system with zero angular viscosity, *Discrete Contin. Dyn. Syst.* **35**(2015), 2193–2207. [MR3294246](#); [url](#)
- [25] B. Q. YUAN, Regularity of weak solutions to magneto-micropolar fluid equations, *Acta Math. Sci. Ser. B Engl. Ed.* **30**(2010), 1469–1480. [MR2778615](#); [url](#)
- [26] X. W. ZHANG, Y. JIA, B. Q. DONG, On the pressure regularity criterion of the 3D Navier-Stokes equations, *J. Math. Anal. Appl.* **393**(2012), 413–420. [MR2891019](#); [url](#)
- [27] Z. J. ZHANG, X. YANG, Remarks on the blow-up criterion for the MHD system involving horizontal components or their horizontal gradients, *Ann. Polon. Math.* **116**(2016), 87–99. [MR3460726](#); [url](#)
- [28] Z. J. ZHANG, X. YANG, Navier–Stokes equations with vorticity in Besov spaces of negative regular indices, *J. Math. Anal. Appl.* **440**(2016), 415–419. [MR3479607](#); [url](#)
- [29] Z. J. ZHANG, Z. A. YAO, X. F. WANG, A regularity criterion for the 3D magneto-micropolar fluid equations in Triebel–Lizorkin spaces, *Nonlinear Anal.* **74**(2011), 2220–2225. [MR2781751](#); [url](#)